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Why Sense-Making through Magnitude May Be Harder for Fractions than for Whole Numbers

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Abstract

What is the role of fraction magnitude knowledge in learning fraction addition? An experiment with 71 6th and 7th grade students compared fraction addition instruction and practice with a magnitude representation to a tightly controlled non-magnitude condition. In the magnitude condition, students with better fraction magnitude estimation skills benefitted more from the conceptual instruction and this relationship was moderated by students' knowledge of how magnitude relates to fraction addition and equivalence. However, students with better fraction magnitude estimation skills benefitted *less* from the practice problems with magnitude. In the non-magnitude condition, fraction magnitude estimation was not predictive of learning. This study indicates that students with magnitude knowledge can leverage it to learn fraction addition concepts from magnitude representations, but, for those students, magnitude representations may be a distraction from practicing the procedure.

Keywords: fraction addition; number line estimation; multiple representations

Fractions are Important but Difficult to Learn

Proficiency with fractions is a pivotal skill for students: fraction knowledge in fifth grade predicts math achievement in tenth grade even after statistically controlling for socioeconomic status, general IQ, and whole number arithmetic knowledge (Siegler et al., 2012). Yet, fractions are a difficult topic for students (Siegler et al., 2010) and teachers (Lee, Brown, & Orrill, 2011; Siegler et al., 2010). One obstacle to students' conceptual understanding is difficulty estimating a fraction's magnitude (Behr, Wachsmuth, & Post, 1985). In addition to improving conceptual understanding, fraction magnitude estimation, specifically accuracy on number line estimation, is correlated with accuracy on fraction arithmetic tasks (Siegler, Thompson, & Schneider, 2011).

One potential reason for this correlation is that students who have stronger fraction number line estimation knowledge can avoid incorrect strategies that result in magnitude incongruent answers. Adding 1/2 and 1/3, for example, must result in an answer greater than 1/2 and 1/3. Students who understand that the magnitude of the sum needs to be greater than the magnitude of both positive addends might reject an incorrect strategy that leads to a magnitude incongruent answer. A common incorrect strategy, adding the numerators and denominators, would lead to a sum of 2/5. Since the magnitude of the incorrect sum (2/5) is smaller than that of one of the addends (1/2) the strategy must be incorrect.

Fifth grade students don't always understand that adding positive fractions results in a sum with a magnitude that is larger than each individual addend (Wiese & Koedinger 2014) while sixth grade students almost always understand this concept (Siegler & Lortee-Forgues, 2015). If the correlation between accuracy of fraction number line estimation and accuracy on fraction arithmetic is at least partially driven by the reduction of magnitude incongruent procedural strategies, the correlation between accuracy of fraction number line estimation and accuracy on fraction arithmetic should be greater for those students who know how addition or subtraction affect magnitude versus those students who do not. That is, students cannot logically use their magnitude knowledge to reject magnitude incongruent answers if they do not know the sum is greater than the addends for positive fractions. Therefore, the understanding of how the magnitude of the addends is related to the sum should moderate the relationship between fraction magnitude knowledge and fraction addition knowledge.

Students also need equivalence magnitude knowledge in addition to fraction magnitude knowledge to help select amongst fraction arithmetic strategies. A student cannot use fraction magnitude knowledge to select amongst fraction addition strategies if she or he does not know that equivalent fractions have the same magnitude. For example, when adding 1/2 and 1/4, students need to find a common denominator (e.g., 2/4 plus 1/4). If students do not know that 1/2 and 2/4 are the same magnitude, it makes it difficult to compare the final sum (3/4) to the original addends. This means equivalence magnitude knowledge should moderate the relationship between fraction magnitude knowledge and fraction addition knowledge. The present study tests the two moderators of the relationship between fraction magnitude knowledge and fraction addition knowledge.

Prior Work: Brief Number Line Interventions

Existing literature supports the positive relationship between magnitude knowledge of whole numbers and whole number arithmetic knowledge (Siegler & Mu, 2008), and the positive, causal influence of whole number magnitude knowledge on whole number arithmetic knowledge with a brief intervention (Booth & Siegler, 2008). Since magnitude-based interventions have been shown to improve students' proficiency whole number arithmetic, they may improve students' proficiency with fraction arithmetic also.

Magnitude Representations May Not Be Sufficient for Fraction Addition Skills

On the other hand, although magnitude training is a promising approach, it may not be sufficient for improving students' arithmetic skills or for their rejection of strategies that ignore magnitude. In one study, 5th grade students were asked to judge if fraction addition equations were correct or not. Half of the equations were correct, and half used the add-both-numerators-and-denominators strategy to yield a sum that was smaller than one of the addends. Even when shown the fraction symbols in the equation and their corresponding magnitudes, students performed around chance (Stampfer & Koedinger, 2013). This result indicates that magnitude was not sufficient for the students to determine the correctness of an addition equation. To reject a strategy that ignores magnitude, students must realize not only that fractions have magnitudes, but also that fractions follow the same rules of addition as whole numbers, namely, that adding two positive fractions yields a sum larger than each addend. A follow-up study confirmed that those 5th grade students had difficulty applying that addend-sum relationship to fractions (Wiese & Koedinger, 2014).

Finally, though choosing an incorrect addition procedure reveals gaps in conceptual knowledge, it also reveals gaps in procedural knowledge. Students may make procedural errors on fraction addition items even though they have high levels of conceptual knowledge. Indeed, Byrnes & Wasik (1991) found, in a an experiment with middle-schoolers, that reinforcing fraction magnitude concepts and explaining the addition procedure did not improve performance above simply executing the procedure. However, Byrnes and Wasik used discrete rectangles instead of a continuous number line. Since there are theoretical reasons a number line specifically might be more conducive to thinking about magnitude (e.g., Siegler et al., 2011) and discrete rectangles might elicit counting behavior as opposed to thinking about magnitude holistically, the present study uses a continuous representation of magnitude.

Study: Fraction Addition Instruction With and Without a Representation of Magnitude

The present study will compare two forms of instruction, one with a magnitude representation and one without. We hypothesize that knowledge and application of the addend-sum relationship (that two positive addends yield a sum larger than each) moderate the relationship between magnitude estimation of fractions and fraction addition skills. Additionally, we propose conflicting hypotheses for the question 'Does a procedure-and-magnitude intervention demonstrate superior learning gains than a procedure-only control?' Magnitude approaches would suggest yes, as considering magnitude should help students reject incorrect answers, while cognitive load theory would suggest no, as the magnitude representations require processing that is extraneous to learning the procedure.

Participants and Assessment Materials

78 6th and 7th graders from two middle schools participated in the study. The analyses include only the 71 students who completed all portions of the study. Within-class random assignment yielded 38 in the procedure-and-magnitude intervention and 33 students in the procedure-only control.

Three assessments were given: before the instruction (pretest), after the instruction (midtest) and after the practice problems (posttest). Three assessment forms were created and students were randomly assigned to one of the six counter-balanced orders. The overall reliabilities (Chronbach's Alpha) for the test forms were .77 or greater.

Fraction Number Line Estimation items presented a target fraction to the left of a number line, which was labeled with a '0' on the left endpoint and a '1' on the right endpoint. Participants saw 10 of these problems, only at pretest.

Students saw four Fraction Addition problems. Two had same denominators while two had unlike denominators. For the addition items, the reliabilities (Chronbach's Alpha) for the three test forms ranged from .72 to .75. The remaining items (described below) assessed how students connected magnitude knowledge to a fraction addition context.

For magnitude knowledge to be beneficial for fraction addition, students will likely need to understand how magnitude relates to addition. Direction of Effects items asked if the addition of two positive fractions is greater than one of those fractions alone (e.g., $1/2 + 1/3 > 1/3$, with answer options True, False, and Can't tell from the information given). Addend-Sum items asked if the sum was greater than each of the addends (e.g., showing students $1/2 + 1/3 = 5/6$ with a note that the addition is correct, then asking if $5/6 > 1/2$ and if $5/6 > 1/3$, with the answer options True, False, and Can't tell from the information given).

For linear depictions of magnitude to be beneficial, students will likely need to be able to interpret them correctly, including recognizing that equivalent fractions have the same magnitudes. Fraction Equivalence items presented a number line with a fraction plotted on it. Students were provided with an equivalent fraction (and the statement that the two fractions were equivalent) and asked to plot the equivalent fraction on the same number line. They were then asked if the plotted fraction belonged to the left, right, or in the same spot as the presented fraction with a fill-in-the-blank multiple-choice question. For example, 2/8 belongs ____ 1/4, with the options "to the left of," "in the exact same spot as," and "to the right of."

Instructional Materials

Students were given initial instruction and then practice problems. First, all students saw an example whole number addition equation, and an explanation that adding whole numbers yields the total of their sizes. A similar example of a fraction addition equation followed, showing that the independent whole number strategy yields an incorrect sum. Accompanying text explained that the correct sum for fraction addition would be the same magnitude as the combined magnitudes of the addends. The instruction included brief multiple-choice and free-response questions to check that students had read the text. This instruction was intended to shake students' confidence in the incorrect whole number strategy, and did not demonstrate the full fraction addition procedure. Students in the procedure-and-magnitude intervention saw depictions of the magnitudes of the addends and sums in the example equations. This depiction was intended to reinforce that a correct sum would equal the combined magnitudes of the addends, while an incorrect sum would not. These depictions were linear magnitude representations, but were not presented on a number line. The procedure-only control did not include depictions of magnitude (see Figure 1). After completing that initial instruction, students did the mid-test.

After the mid-test, students did scaffolded practice problems with correctness feedback for each step, alternating with non-scaffolded problems (Figure 2), which were similar to the fraction addition assessment items. After entering an answer for the non-scaffolded problems, students were given correctness feedback. If the answer was wrong, students were shown a worked example of that problem and were asked to fix their answer. The worked examples in the procedure-and-magnitude intervention included a linear magnitude depiction of the two original addends, the student's incorrect answer, and the correct answer. Students in the procedure-only control did the same tasks, but without the depictions of magnitude.

Method

All study activities took place during normal class time. Students were given 2 minutes for the number-line estimation questions, 5 for the pre-test and post-test, and 25 minutes for the instruction, mid-test, and practice problems, which students did at their own pace. All activities used an online math tutoring system. While the instruction between the pre-test and mid-test was brief, it targeted only one concept: that a correct addition strategy will yield a sum that is equal in magnitude to the two addends together. After the mid-test, students practiced fraction addition with scaffolding, interleaved with worked examples. This study was timed for a standard 40-minute class period. Though students would likely have improved more over a longer study duration, previous work has demonstrated learning with brief interventions that include magnitude (Lomas, Ching, Stampfer, Sandoval, & Koedinger, 2011; Opfer & Thompson, 2008; Siegler & Ramani, 2009).

Results

For the fraction addition items, students rated their answers with a confidence score from 1 to 5. The answer score

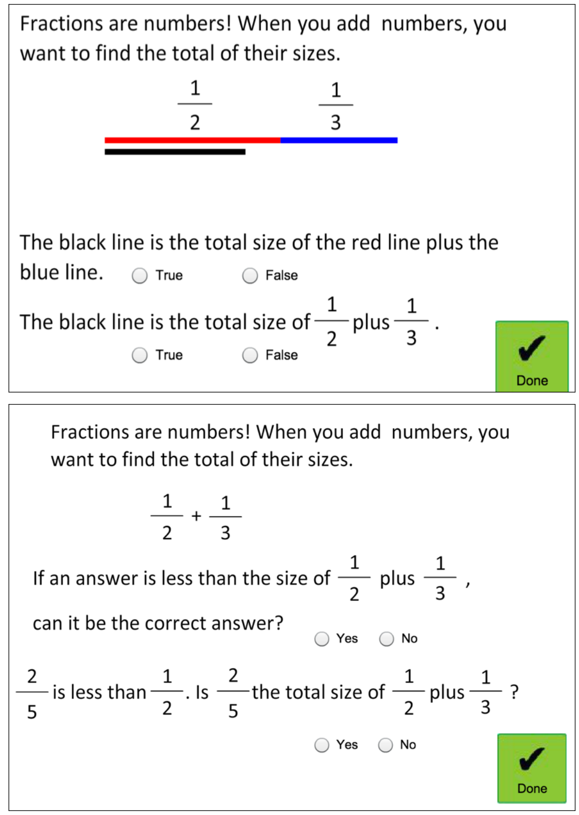

Figure 1: Matched instruction with magnitude (top) and without it (bottom), question four of six. Both formats try to convey that a sum cannot be smaller than the combined magnitudes of its addends.

(correct or incorrect) was combined with the self-reported accuracy to yield a scale from 1 to 10 (from highest confidence in an incorrect answer to highest confidence in a correct answer). This measure is referred to below as 'accuracy' (Note: Similar patterns of results were obtained using accuracy without confidence). Students' responses spanned the entire scale. Table 2 shows means and standard deviations for fraction addition accuracy, and mean scores for direction of effects problems, addend sum problems, and the multiple-choice portion of the fraction equivalence problems. For the latter three items, since there was only one question per student per type of problem, the distributions are binomial and can be expressed in a single parameter.

One striking finding from this study was students' difficulty in progressing through the initial instruction (between the pre-test and mid-test), which was intended to be straightforward. This initial instruction presented six problems, starting with whole number addition and moving to fraction addition, all focused on the idea that the sum of two numbers is the total of their magnitude. In particular, the error rate for question four (Figure 1) reveals gaps in students' reasoning around fraction addition. In the procedure-only control, the majority of students (85%) answered at least one question incorrectly, even though the

Table 2: Assessment Scores

Fraction Addition Accuracy Means (and Std. Deviations)			
Condition	Pretest	Midtest	Posttest
Procedure-only	5.22 (2.81)	5.45 (2.84)	6.48 (2.30)
Procedure & Magnitude	5.14 (2.58)	5.26 (2.61)	6.35 (2.64)
Direction of Effects Mean Scores			
Condition	Pretest	Midtest	Posttest
Procedure-only	45%	61%	55%
Procedure & Magnitude	42%	38%	42%
Addend-Sum Mean Scores			
Condition	Pretest	Midtest	Posttest
Procedure-only	56%	58%	56%
Procedure & Magnitude	55%	54%	57%
Equivalence Mean Scores			
Condition	Pretest	Midtest	Posttest
Procedure-only	21%	21%	13%
Procedure & Magnitude	29%	30%	27%

first question explicitly states that the proposed sum is less than combined sizes of the two addends, and the second question explicitly states that the proposed sum is less than the size of one of the addends alone. While the matched questions were easier in the procedure-and-magnitude intervention (which showed the magnitude of each fraction graphically), students were still far from ceiling (34% had at least one incorrect response). One hypothesis for why students benefit from magnitude knowledge is that it helps them reject magnitude-incongruent answers. That hypothesis suggests that knowledge of magnitude helps students arrive at the correct evaluation of whether or not a sum is incongruent. However, this information was explicitly provided in the procedure-only control, and students' error rates show that version of the question was harder. These results suggest that it is the underlying addition principles that are hard for students to make sense of in a fraction addition context. The hypothesis that magnitude knowledge helps students by making it easier for them to apply principles of addition presupposes that students understand the principles of addition in the first place. Rather, these results suggest the reverse: depictions of magnitude may help students make sense of abstract principles of addition.

Analyses

Across both conditions, a pairwise t-test revealed no difference between pretest fraction addition scores and mid-test fraction addition scores (t = -0.46, df = 68, p = 0.65). In a linear regression, pretest addition scores and condition were predictive of mid-test scores ($F(2,66) = 23.29$, R^2 = 41.37%, $p < 0.01$). While pretest fraction addition accuracy was predictive of mid-test fraction addition accuracy (B = 0.65, t = 6.814, $p < 0.01$), condition was not predictive (B = -0.20, t = -0.39, p = 0.70).

Within each condition, paired t-tests with pretest fraction addition accuracy and mid-test fraction addition accuracy

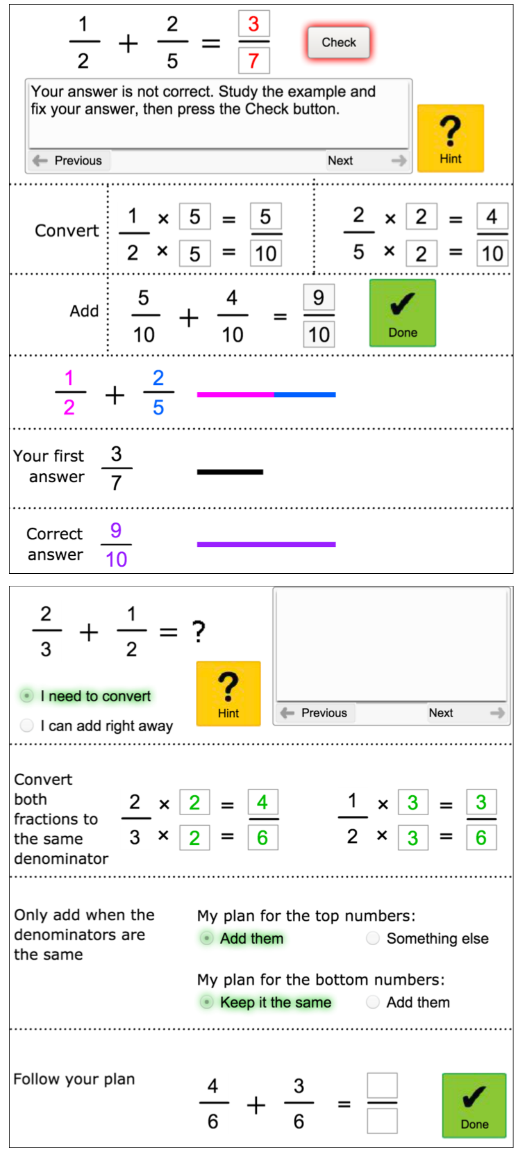

Figure 2: A non-scaffolded problem (top) and scaffolded problem (bottom). Non-scaffolded items initially show only the addition problem. If the student answers incorrectly, the worked example is revealed. Scaffolded items unfold step by step, and are the same in both conditions.

reveal no significant improvement (magnitude: t = -0.07, df = 36, p = 0.94; control: t = -0.75, df = 31, p = 0.46). Within each condition, linear regressions predicted mid-test fraction addition accuracy with pretest fraction addition accuracy and Percent Average Error (PAE) for fraction number line estimation (magnitude: $F(2,34) = 12.63$, R^2 = 42.62%, $p <$ 0.01; control: $F(2,29) = 27.68$, R^2 = 65.62%, $p < 0.01$). In the procedure-and-magnitude invervention, both pretest fraction addition accuracy (B = 0.29, t = 2.01, p = 0.05) and PAE for fraction number line estimation (B = -11.01, t = -3.47, $p < 0.01$) were significant predictors of mid-test fraction addition accuracy. Note that a negative predictor for PAE means low error (i.e., high accuracy) in number line estimation is associated with higher scores on the fraction addition accuracy scale at mid-test. Since there was no learning in fraction addition from pre-test to mid-test and PAE was a significant predictor, it seems that students got better at fraction addition if they had strong magnitude knowledge, but got worse if they had weak magnitude knowledge. For the procedure-only control, pretest fraction addition accuracy was a significant predictor of mid-test fraction addition accuracy ($B = 0.81$, $t = 7.33$, $p < 0.01$), but PAE was not ($B = -1.04$, $t = -0.45$, $p = 0.66$). The two groups seem to have learned in different ways: students' number line magnitude knowledge influenced their learning for the procedure-and-magnitude intervention, but not for the procedure-only control.

Three separate regressions on the procedure-and-magnitude intervention tested moderators of fraction magnitude knowledge and fraction addition learning from pre-test to mid-test: scores on direction of effects, equivalence, and addend-sum items. Separate regressions were run since the moderators were correlated, violating the independent and identical distribution assumption (i.i.d.) of regression.

Moderators: PAE and Direction of Effects This regression included pretest fraction addition accuracy, PAE fraction number line estimation, pretest direction of effects accuracy, and the interaction between PAE and pretest direction of effects accuracy ($F(4,32) = 9.414$, $R^2 = 54.06\%$, $p < 0.01$). In this model, pretest fraction addition accuracy is not predictive of mid-test fraction addition accuracy ($B = 0.19$, $t = 1.40$, $p = 0.17$), but PAE ($B = -8.82$, $t = -2.83$, $p < 0.01$), pretest direction of effects score ($B = 1.43$, $t = 2.070$, $p = 0.05$), and the interaction of the two ($B = -0.60$, $t = -1.66$, $p = 0.10$) are predictive. Thus, for fraction addition accuracy, students with low magnitude error and high direction of effect accuracy benefited the most from the procedure-and-magnitude intervention.

Moderators: PAE and Equivalence Knowledge The second regression uses pretest fraction addition accuracy, PAE fraction number line estimation, pretest equivalence knowledge, and the interaction between PAE and pretest equivalence knowledge to predict mid-test fraction addition accuracy ($F(4,32) = 8.996$, $R^2 = 52.93\%$, $p < 0.01$). Pretest fraction addition accuracy ($B = 0.12$, $t = 0.84$, $p = 0.41$) and pretest equivalence knowledge ($B = -1.17$, $t = -0.91$, $p = 0.37$) were not predictive of mid-test fraction addition accuracy, but PAE ($B = -14.81$, $t = -3.51$, $p < 0$ .01) and the interaction between PAE and pretest equivalence knowledge ($B = -1.48$, $t = -2.32$, $p = 0.03$) are predictive of mid-test fraction addition accuracy. Thus, for fraction addition accuracy, students with low magnitude error and high equivalence accuracy benefited most from the intervention.

Moderators: PAE and Addend-Sum The final regression included pretest fraction addition accuracy, PAE fraction number line estimation, addend sum knowledge questions, and the interaction between PAE and addend sum knowledge to predict mid-test fraction addition accuracy ($F(4,31) = 6.097$, $R^2 = 44.03\%$, $p < 0.01$. Pretest fraction addition accuracy ($B = 0.30$, $t = -3.10$, $p = 0.06$) and PAE ($B = -10.49$, $t = -3.104$, $p < 0.01$) were both predictive of mid-test fraction addition accuracy, but addend sum knowledge ($B = 0.00$, $t = 0.00$, $p > 0.99$) and the interaction between PAE and addend sum knowledge ($B = -0.36$, $t = -0.97$, $p = 0.34$) were not predictive of mid-test fraction addition accuracy.

Midtest to Posttest Learning Across both conditions, a pairwise t-test revealed improvement in fraction addition scores from mid-test to post-test ($t = -3.81$, $df = 66$, $p < 0.01$; means were 5.35 at mid-test vs 6.41 at post-test). In a linear regression, mid-test addition scores and condition were predictive of post-test scores ($F(2,64) = 26.38$, $R^2 = 45.19\%$, $p < 0.01$). While mid-test fraction addition accuracy was predictive of post-test fraction addition accuracy ($B = 0.62$, $t = 7.258$, $p < 0.01$), condition was not predictive ($B = -0.11$, $t = -0.24$, $p = 0.81$).

Midtest to Posttest Learning and PAE For both conditions individually, paired t-tests on mid-test and post-test fraction addition accuracy show learning (magnitude: $t = -2.92$, $df = 34$, $p < 0.01$; control: $t = -2.48$, $df = 31$, $p = 0.02$). In a linear regression with the procedure-and-magnitude invervention only, post-test fraction addition accuracy was predicted using mid-test fraction addition accuracy and PAE for fraction number line estimation ($F(2,32) = 24.05$, $R^2 = 60.05\%$, $p < 0.01$). Both mid-test fraction addition accuracy ($B = 0.92$, $t = 6.54$, $p < 0.01$) and PAE ($B = 6.37$, $t = 2.01$, $p < 0.01$) were significant predictors of post-test fraction addition accuracy. Note that a positive predictor for PAE here means low error (or high accuracy) in number line estimation is associated with *lower* scores on the post-test fraction addition accuracy measure. When running the same regression with the procedure-only control ($F(2,29) = 8.49$, $R^2 = 36.92\%$, $p < 0.01$), mid-test fraction addition accuracy was a significant predictor post-test fraction addition accuracy ($B = 0.48$, $t = 3.95$, $p < 0.01$), but PAE was not ($B = -1.56$, $t = -0.62$, $p = 0.54$). Note: We tested the same moderators we used from pretest to mid-test, but none were significant.

Discussion

What is the role of fraction magnitude knowledge in learning fraction arithmetic? Within the procedure-and-magnitude intervention, when students were learning how magnitude knowledge can be used to justify correct and incorrect answers (between pretest and mid-test), students with greater initial magnitude knowledge benefited more than students with lower initial magnitude knowledge. However, when solving the problems and studying the worked examples (between mid-test and post-test), students with greater initial magnitude knowledge benefitted *less* than their counterparts. Perhaps the magnitude knowledge that was initially helpful for learning the general concept of fraction addition became a distractor to learning the procedural steps. Perhaps the combination of learning the right steps to solving a fraction addition problem *and* seeing how the magnitudes of their answers compared to the magnitude of the actual answer exceeded the desirable difficulty. Trying to learn both pieces of information at once (each of which can be considered complex) may have caused excessive cognitive load and hurt their performance. Those with stronger magnitude knowledge might have been more distracted by the magnitude stimuli if they were attending to it more, while those with weaker magnitude knowledge might have attended to the magnitude stimuli less, resulting in more focus in the steps, and thus, better performance.

Also of note is the lack of differences between the groups in terms of overall learning. While the mechanisms by which the students learned might have been different, both groups overall did not improve from pretest to midtest and learned equally from midtest to posttest. This indicates the benefits of learning by doing: overall, each condition only demonstrated learning from the activities that gave students practice solving the targeted problems. The results from this experiment do not provide evidence supporting the hypothesis that the inclusion of magnitude in a brief fraction addition instruction benefits students' learning. However, the role of magnitude knowledge in the procedure-and-magnitude interention does provide evidence for the theoretical role of magnitude in fraction addition learning: explicit representations of fraction magnitude can help students learn fraction concepts but may detract from learning the procedures.

Finally, the initial instruction (between pre-test and mid-test) posed surprising challenges for the students. Students demonstrated difficulty in making logical inferences involving fraction addition. For example, 85% of students in the procedure-only control thought that an answer that was less than the total size 1/2 and 1/3 could be the correct answer to 1/2 + 1/3. While the correct inference was easier to make in the procedure-and-magnitude intervention, where the magnitudes of the addends and proposed sum were provided, 34% of students in that condition still made errors. Results from students' interactions with the initial instruction indicate that students did not have a solid, context-general foundation with the principles of addition.

It is important to consider how this type of instruction might generalize to other fraction arithmetic operations. While there is evidence for the mechanistic use of magnitude knowledge, in this study there was little practical benefit of the procedure-and-magnitude intervention compared to the control for learning the fraction addition procedure.

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